Equivariant spectral triple for the compact quantum group $U_q(2)$ for complex deformation parameters

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(joint work with Bipul Saurabh)

Background

Let $\mathbb{G}$ be a compact quantum group acting on a C^*-algebra $\mathbb{A}$ via the action $\tau : \mathbb{A} \longrightarrow \mathbb{A} \otimes C(\mathbb{G})$, so that we have a C^*-dynamical system $(\mathbb{A}, \mathbb{G}, \tau)$. Let $(\pi, \mathbb{U})$ be a covariant representation of the C^*-dynamical system $(\mathbb{A}, \mathbb{G}, \tau)$ on a Hilbert space $\mathcal{H}$. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$ be an even spectral triple on $\mathbb{A}$.

1. The Dirac operator $\mathcal{D}$ is called "equivariant" under the action τ if $\mathcal{D} \otimes 1$ commutes with $\mathbb{U}$.
2. If $\gamma \otimes 1$ also commutes with $\mathbb{U}$, then the spectral triple is called $\mathbb{G}$-equivariant.

Taking $F = \mathcal{D}|\mathcal{D}|^{-1}$, the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$ induces a K-homology class $[(\mathcal{A}, \mathcal{H}, F, \gamma)]$ in $K^0(\mathbb{A})$ consisting of even Fredholm modules. To check nontriviality of this class, one pairs it with $K_0(\mathbb{A})$ through the Kasparov product.

Given a projection $P \in M_n(\mathbb{A})$ define

$$\mathcal{H}_n = \mathcal{H} \otimes \mathbb{C}^n \quad , \quad \gamma_n = \gamma \otimes I_n \quad , \quad F_n = F \otimes I_n \, ,$$

$$P^+ = \tfrac{1+\gamma_n}{2} P \quad , \quad P^- = \tfrac{1-\gamma_n}{2} P \, .$$

The following operator

$$P^- F_n P^+ : P^+ \mathcal{H}_n^+ \longrightarrow P^- \mathcal{H}_n^-$$

is a Fredholm operator, where $\mathcal{H}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$ under the grading operator γ_n. Index of this Fredholm operator is the value of the $K_0 - K^0$ pairing $\langle [P], [(\mathcal{A}, \mathcal{H}, F, \gamma)] \rangle$.

1. An even spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$ is called nontrivial if $\langle [P], [(\mathcal{A}, \mathcal{H}, F, \gamma)] \rangle$ is nonzero for some $[P] \in K_0(\mathbb{A})$.

Motivation

A compact quantum group $\mathbb{G}$ acts on its underlying C^*-algebra $\mathbb{A} = C(\mathbb{G})$ via the comultiplication Δ, and we have the C^*-dynamical system $(C(\mathbb{G}), \mathbb{G}, \Delta)$. A natural choice for $\mathcal{A}$ is the dense Hopf $\star$-subalgebra $\mathcal{O}(\mathbb{G})$ generated by the matrix coefficients of f.d. corepresentations of $(C(\mathbb{G}), \Delta)$.

Now, one can desire to produce an explicit $\mathbb{G}$-equivariant nontrivial Dirac operator on $\mathcal{A}$. We are interested in compact quantum groups arising from the semidirect product construction of Woronowicz et al., for which $U_q(2)$ is a nontrivial concrete example.

Origin of $U_q(2)$

In (Kasprzak-Meyer-Roy-Woronowicz, 2016), Woronowicz et al. defined a family of q-deformations of $SU(2)$ for $q \in \mathbb{C}^*$. This agrees with the compact quantum group $SU_q(2)$ when q is real. But for $q \in \mathbb{C} \setminus \mathbb{R}$, $SU_q(2)$ is not a compact quantum group, rather a braided compact quantum group. In (Meyer-Roy-Woronowicz, 2016), Woronowicz et al. showed that for a compact quantum group $\mathbb{G} = (\mathbb{A}, \Delta)$ and a braided compact quantum group $\mathbb{B}$ over $\mathbb{G}$, the semidirect product $\mathbb{A} \boxtimes \mathbb{B}$ becomes a compact quantum group.

Taking $\mathbb{B} = SU_q(2)$ for $q \in \mathbb{C}^*$ and $\mathbb{A} = C(\mathbb{T})$, we obtain a genuine compact quantum group, and it is the coopposite of the compact quantum group $U_q(2)$ defined in (Zhang-Zhao, 2005).

Definition: The CQG $U_q(2)$

Let $q \in \mathbb{C}^*$. The C^*-algebra $C(U_q(2))$, to be denoted by $\mathbb{A}_q$ throughout, is the universal C^*-algebra generated by a, b, D satisfying the following relations :

$$\begin{aligned} ba &= qab, \quad a^*b = qba^*, \quad bb^* = b^*b, \quad aa^* + bb^* = 1, \\ aD &= Da, \quad bD = q^2|q|^{-2}Db, \quad DD^* = D^*D = 1, \quad a^*a + |q|^2b^*b = 1. \end{aligned}$$

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The compact quantum group structure is given by the comultiplication $\Delta : \mathbb{A}_q \longrightarrow \mathbb{A}_q \otimes \mathbb{A}_q$ defined as follows :

$$\Delta(a) = a \otimes a - \bar{q}b \otimes Db^* \;,\; \Delta(b) = a \otimes b + b \otimes Da^* \;,\; \Delta(D) = D \otimes D.$$

Let $\mathcal{O}(U_q(2))$ be the $\star$-subalgebra of $C(U_q(2))$ generated by a, b and D. The Hopf $\star$-algebra structure on it is given by the following :

$$\begin{aligned} \text{antipode:} \quad & S(a) = a^*,\; S(b) = -qbD^*,\; S(D) = D^*, \\ & S(a^*) = a,\; S(b^*) = -(\bar{q})^{-1}b^*D\,, \\ \text{counit:} \quad & \epsilon(a) = 1,\; \epsilon(b) = 0,\; \epsilon(D) = 1\,. \end{aligned}$$

Faithful representations of $C(U_q(2))$ for $|q| < 1$

We concentrate on the case of $|q| \neq 1$. It is enough to restrict our attention to the case $|q| < 1$ and $q \neq 0$ because $U_q(2)$ and $U_{\frac{1}{\bar{q}}}(2)$ are isomorphic as CQGs.

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Fix any $q \in \mathbb{C}^*$ with $|q| < 1$ and let $\theta = \frac{1}{\pi} \arg(q)$. Let $\mathcal{H}$ be the Hilbert space $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$. Consider the right shift $V : e_n \mapsto e_{n+1}$ acting on $\ell^2(\mathbb{N})$ and the bilateral shift $U : e_n \mapsto e_{n+1}$ acting on $\ell^2(\mathbb{Z})$.
Define the following representation π of $\mathbb{A}_q$ on $\mathcal{H}$:

$$\pi(a) = \sqrt{1 - |q|^{2N}}\, V \otimes 1 \otimes 1 \quad , \quad \pi(b) = q^N \otimes U \otimes 1 \, ,$$
$$\pi(D) = 1 \otimes e^{-2\pi\sqrt{-1}\theta N} \otimes U \, .$$

Proposition

The representation π of $\mathbb{A}_q$ defined above is faithful.

Linear basis and the Haar state

Define

$$\langle n,m,k,l\rangle = \begin{cases} a^n b^m (b^*)^k D^l & \text{if } n \geq 0, \\ (a^*)^{-n} b^m (b^*)^k D^l & \text{if } n \leq 0. \end{cases}$$

Theorem

The set $\{\langle n,m,k,l\rangle : n,l \in \mathbb{Z}, m,k \in \mathbb{N}\}$ *is a linear basis of* $\mathcal{O}(U_q(2))$.

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Theorem

The Haar state $h : C(U_q(2)) \longrightarrow \mathbb{C}$ *is given by the following,*

$$h(x) = (1 - |q|^2) \sum_{n=0}^{\infty} |q|^{2n} \langle e_{n,0,0}\, , \, \pi(x) e_{n,0,0}\rangle \, ,$$

where $\{e_{n,r,s}\}$ *denotes the standard orthonormal basis of* $\ell^2(\mathbb{N} \times \mathbb{Z} \times \mathbb{Z})$.

Torus quotient

Moreover, one has

$$h(\langle n,m,k,l\rangle) = \begin{cases} \frac{1-|q|^2}{1-|q|^{2(m+1)}} & \text{if } m=k, \text{ and } n=l=0, \\ 0 & \text{otherwise .} \end{cases}$$

In this case of $|q|<1$, the Haar state is not a trace as

$$h(a^*a - aa^*) = (1-|q|^2)h(bb^*) = \tfrac{1-|q|^2}{1+|q|^2} \neq 0\,.$$

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Let $\Delta_{\mathbb{T}}$ denotes the coproduct on $C(\mathbb{T})$ and $\phi : \mathbb{A}_q \longrightarrow C(\mathbb{T})$ be the homomorphism given by $\phi(a)=1$, $\phi(b)=0$ and $\phi(D)=\mathfrak{z}$. It follows that $\Delta_{\mathbb{T}} \circ \phi = (\phi\otimes\phi)\circ\Delta$. Thus, $\mathbb{T}$ is a quantum subgroup of $U_q(2)$ and hence, $\mathbb{T}$ acts on $\mathbb{A}_q$ by the formula $\Phi(x) = (\mathrm{id}\otimes\phi)\Delta$. In such a case, one defines the quotient space $U_q(2)/\mathbb{T}$ as follows,

$$C(U_q(2)/\mathbb{T}) = \{x\in\mathbb{A}_q : (\mathrm{id}\otimes\phi)\Delta(x) = x\otimes 1\}.$$

Faithfulness of the Haar state

The conditional expectation $E : \mathbb{A}_q \longrightarrow C(U_q(2)/\mathbb{T})$ is defined as $(\mathrm{id} \otimes (h_{\mathbb{T}} \circ \phi)) \circ \Delta$, where $h_{\mathbb{T}}$ denotes the Haar state on $\mathbb{T}$.

Proposition

The C^*-algebra $C(U_q(2)/\mathbb{T})$ is the C^*-subalgebra of $\mathbb{A}_q$ generated by a and b, and $C(U_q(2)/\mathbb{T}) = C(SU_{|q|}(2))$.

Theorem

The Haar state on the compact quantum group $U_q(2)$ is faithful.

Corepresentations of $\mathcal{A}_q$

For $\ell \in \frac{1}{2}\mathbb{N}$ and $i, j \in I_\ell := \{-\ell, \cdots\cdots, \ell\}$, let

$$f_j^\ell = \binom{2\ell}{\ell + j}_{|q|^2}^{1/2} a^{\ell - j} b^{\ell + j},$$
$$V_\ell^R = \oplus_{j=-\ell}^{\ell} \mathbb{C} f_j^\ell \quad \text{and} \quad T_\ell^R = \Delta|_{V_\ell^R}.$$

That is, V_ℓ^R is the vector subspace of $\mathcal{A}_q$ spanned by degree 2ℓ-homogeneous monomials in a and b. We get a corepresentation $T_\ell^R : V_\ell^R \longrightarrow V_\ell^R \otimes \mathcal{A}_q$ of $\mathcal{A}_q$ on V_ℓ^R. For $i, j \in I_\ell$, let t_{ij}^ℓ denote the matrix coefficients of T_ℓ^R with respect to the basis $\{f_j^\ell\}$ of V_ℓ^R, and we have

$$T_\ell^R(f_j^\ell) = \Delta(f_j^\ell) = \sum_{i=-\ell}^{\ell} f_i^\ell \otimes t_{ij}^\ell .$$

Hence, T_ℓ^R is a $(2\ell + 1)$-dimensional corepresentation of $\mathcal{A}_q$.

Moreover, there are infinitely many one dimensional corepresentations D^m of $\mathcal{A}_q$.

Consider the following sesquilinear forms $\langle\cdot,\cdot\rangle_R$ and $\langle\cdot,\cdot\rangle_L$ on $\mathcal{A}_q$,

$$\langle x,y\rangle_L = h(x^*y) \quad , \quad \langle x,y\rangle_R = \overline{h(xy^*)} \qquad \text{for } x,y\in\mathcal{A}_q\,.$$

In the Hilbert space $(V_\ell^R,\langle\cdot,\cdot\rangle_L)$, the set $B_\ell := \left\{|q|^\ell|2\ell+1|_{|q|}^{1/2} f_j^\ell\right\}_{i\in I_\ell}$ is an orthonormal basis of V_ℓ^R.

Theorem

For $m\in\mathbb{Z}$ and $\ell\in\frac{1}{2}\mathbb{N}$, $T_\ell^R D^m$ is an irreducible corepresentation of $\mathcal{A}_q$ and its matrix coefficients with respect to the basis B^ℓ are $\{t_{ij}^\ell D^m : i,j\in I_\ell\}$. Moreover, the matrix $\left((t_{ij}^\ell D^m)\right)$ is a unitary element of $M_{2\ell+1}(\mathbb{C})\otimes\mathcal{A}_q$.

The Peter-Weyl decomposition

Theorem

Given a corepresentation T of $\mathcal{A}_q$, let $\mathcal{C}(T)$ denotes the vector subspace of $\mathcal{A}_q$ generated by the matrix coefficients of T. Then, we have

1. $\mathcal{A}_q = \bigoplus_{\ell \in \frac{1}{2}\mathbb{N}, m \in \mathbb{Z}} \quad \mathcal{C}(T_\ell^R D^m)$.
2. $$\begin{aligned} \langle t_{ij}^\ell D^m, t_{i^{'}j^{'}}^{\ell^{'}} D^{m^{'}} \rangle_R &= |q|^{-2j} |2\ell+1|_{|q|}^{-1} \delta_{\ell\ell'} \delta_{ii'} \delta_{jj'} \delta_{mm'} \\ \langle t_{ij}^\ell D^m, t_{i^{'}j^{'}}^{\ell^{'}} D^{m^{'}} \rangle_L &= |q|^{2i} |2\ell+1|_{|q|}^{-1} \delta_{\ell\ell'} \delta_{ii'} \delta_{jj'} \delta_{mm'} \end{aligned}$$
3. *The set $\{T_\ell^R D^m : \ell \in \frac{1}{2}\mathbb{N}, m \in \mathbb{Z}\}$ is a complete list of irreducible mutually inequivalent corepresentations of $\mathcal{A}_q$.*
4. *The set $\left\{|q|^{-i}|2\ell+1|_{|q|}^{1/2} t_{ij}^\ell D^m : \ell \in \frac{1}{2}\mathbb{N}, m \in \mathbb{Z}\right\}$ is an orthonormal basis of $L^2(h)$.*

Little q-Jacobi polynomials and matrix coefficients

Theorem

The matrix coefficients $t^{\ell}_{ij}D^k$ are expressed in terms of the little q-Jacobi polynomials in the following way :

1. *for the case of* $i+j \leq 0$, $i \geq j$,
$$a^{-(i+j)}c^{i-j}(\bar{q})^{(j-i)(\ell+j)}\frac{\binom{2\ell}{\ell+j}^{1/2}_{|q|^2}}{\binom{2\ell}{\ell+i}^{1/2}_{|q|^2}}\binom{\ell-j}{i-j}_{|q|^2}\mathcal{P}^{(i-j,-i-j)}_{\ell+j}(bb^*;|q|^2)D^{\ell+j+k};$$
2. *for the case of* $i+j \leq 0$, $i \leq j$,
$$a^{-(i+j)}b^{j-i}q^{(i-j)(\ell+i)}\frac{\binom{2\ell}{\ell+j}^{1/2}_{|q|^2}}{\binom{2\ell}{\ell+i}^{1/2}_{|q|^2}}\binom{\ell+j}{j-i}_{|q|^2}\mathcal{P}^{(j-i,-i-j)}_{\ell+i}(bb^*;|q|^2)D^{\ell+i+k};$$
3. *for the case of* $i+j \geq 0$, $i \leq j$,
$$q^{(i-j)(\ell+i)}\frac{\binom{2\ell}{\ell+j}^{1/2}_{|q|^2}}{\binom{2\ell}{\ell+i}^{1/2}_{|q|^2}}\binom{\ell+j}{j-i}_{|q|^2}\mathcal{P}^{(j-i,i+j)}_{\ell-j}(bb^*;|q|^2)(a^*)^{i+j}b^{j-i}D^{\ell+i+k};$$

Tensor product decomposition and classification

1. for the case of $i + j \geq 0$, $i \geq j$,

$$(\bar{q})^{(j-i)(\ell+j)} \frac{\binom{2\ell}{\ell+j}_{|q|^2}^{1/2}}{\binom{2\ell}{\ell+i}_{|q|^2}^{1/2}} \binom{\ell-j}{i-j}_{|q|^2} \mathcal{P}_{\ell-i}^{(i-j,i+j)}(bb^*;|q|^2)(a^*)^{i+j} c^{i-j} D^{\ell+j+k} .$$

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Theorem

The following decomposition holds,

$$\begin{aligned} & T_{\ell_1} D^m \otimes T_{\ell_2} D^n \\ \simeq\ & T_{\ell_1+\ell_2} D^{m+n} \oplus T_{(\ell_1+\ell_2)-1} D^{m+n+1} \oplus \cdots \oplus T_{|\ell_1-\ell_2|} D^{m+n+2\min\{\ell_1,\ell_2\}}. \end{aligned}$$

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Theorem

Let q and q' be two non-zero complex numbers which are not roots of unity. Then, $U_q(2)$ and $U_{q'}(2)$ are isomorphic as CQGs if and only if $q' \in \{q, \bar{q}, \frac{1}{q}, \frac{1}{\bar{q}}\}$.

Towards the K-groups

For $q = |q|e^{i\pi\theta}$, let $\theta \notin \mathbb{Q} \setminus \{0, 1\}$. Let $\mathscr{T} := C^*(V)$ be the Toeplitz algebra. We have the well-known short exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{\iota} \mathscr{T} \xrightarrow{\sigma} C(\mathbb{T}) \longrightarrow 0$$

where $\sigma : V \longmapsto \mathfrak{z}$. Consider the homomorphism

$$\tau : C(U_q(2)) \longrightarrow C(\mathbb{T}) \otimes \mathcal{B}(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}))$$

given by $\tau = \sigma \otimes 1 \otimes 1$, and for $b_0 = p \otimes U \otimes 1$ let

$$\mathcal{I}_\theta = \text{ the closed two-sided ideal of } C(U_q(2)) \text{ generated by } b_0 \text{ and } b_0^*,$$
$$\mathcal{B}_\theta = C^*\big(\{\tau(a_0)\,,\,\tau(D_\theta)\}\big) = C^*\big(\{\mathfrak{z} \otimes 1 \otimes 1\,,\, 1 \otimes e^{-2\sqrt{-1}\pi\theta N} \otimes U\}\big)\,.$$

Proposition

The following chain of C^*-algebras

$$0 \longrightarrow \mathcal{I}_\theta \xrightarrow{\iota} C(U_q(2)) \xrightarrow{\tau} \mathcal{B}_\theta \longrightarrow 0$$

is an exact sequence, where 'ι' denotes the inclusion map.

The K-groups

Lemma

Let $\mathbb{A}_\theta$ be the noncommutative torus. Then, $\mathcal{I}_\theta = \mathcal{K}(\ell^2(\mathbb{N})) \otimes \mathbb{A}_\theta$.

Theorem

For $q = |q|e^{\sqrt{-1}\pi\theta}$ with θ irrational, both the K-groups $K_0(C(U_q(2)))$ and $K_1(C(U_q(2)))$ are isomorphic to $\mathbb{Z}^2$. The equivalence classes of unitaries $[D]$ and $[p \otimes U \otimes 1 + (1-p) \otimes 1 \otimes 1]$ form a $\mathbb{Z}$-basis for $K_1(C(U_q(2)))$. The equivalence classes of projections $[1]$ and $[p \otimes p_\theta]$ form a $\mathbb{Z}$-basis for $K_0(C(U_q(2)))$, where p_θ denotes the Powers-Rieffel projection with trace θ in the noncommutative torus $\mathbb{A}_\theta$, and $p = |e_0\rangle\langle e_0|$.

Action of the generators on $L^2(h)$

Let us fix the following notation

$$\begin{aligned} e^{\ell}_{i,j,k} &:= |q|^{-i}\sqrt{|2\ell+1|_{|q|}}\, t^{\ell}_{i,j}(D^*)^k \\ &= |q|^{-i}\sqrt{|2\ell+1|_{|q|}}\, t^{\ell}_{i,j}D^{-k} \end{aligned}$$

for the orthonormal basis of $L^2(h)$. Here, $k \in \mathbb{Z}$, $\ell \in \frac{1}{2}\mathbb{N}$ and $i, j \in \{-\ell, \cdots, \ell\}$.

Theorem

The action of the generators of $U_q(2)$ on the orthonormal basis element $e^{\ell}_{i,j,k}$ is described by the following :

1. $D \triangleright e^{\ell}_{i,j,k} = \left(\frac{q}{\bar{q}}\right)^{(i-j)} e^{\ell}_{i,j,k-1}$
2. $D^* \triangleright e^{\ell}_{i,j,k} = \left(\frac{\bar{q}}{q}\right)^{(i-j)} e^{\ell}_{i,j,k+1}$

Theorem

1. $b \triangleright e^{\ell}_{i,j,k} = \beta_+(\ell,i,j)e^{\ell+1/2}_{i-1/2\,,\,j+1/2\,,\,k} + \beta_-(\ell,i,j)e^{\ell-1/2}_{i-1/2\,,\,j+1/2\,,\,k-1}$
2. $b^* \triangleright e^{\ell}_{i,j,k} = \beta^+_+(\ell,i,j)e^{\ell+1/2}_{i+1/2\,,\,j-1/2\,,\,k+1} + \beta^+_-(\ell,i,j)e^{\ell-1/2}_{i+1/2\,,\,j-1/2\,,\,k}$
3. $a \triangleright e^{\ell}_{i,j,k} = \alpha_+(\ell,i,j)e^{\ell+1/2}_{i-1/2\,,\,j-1/2\,,\,k} + \alpha_-(\ell,i,j)e^{\ell-1/2}_{i-1/2\,,\,j-1/2\,,\,k-1}$
4. $a^* \triangleright e^{\ell}_{i,j,k} = \alpha^+_+(\ell,i,j)e^{\ell+1/2}_{i+1/2\,,\,j+1/2\,,\,k+1} + \alpha^+_-(\ell,i,j)e^{\ell-1/2}_{i+1/2\,,\,j+1/2\,,\,k}$

where,

$$\beta_+(\ell,i,j) = q^{\ell-j}\sqrt{\frac{(1-|q|^{2(\ell+j+1)})(1-|q|^{2(\ell-i+1)})}{(1-|q|^{2(2\ell+1)})(1-|q|^{2(2\ell+2)})}}.$$

$$\beta_-(\ell,i,j) = -q^{\ell-j-1}(\bar{q})^{j-i+1}\sqrt{\frac{q}{\bar{q}}}\sqrt{\frac{(1-|q|^{2(\ell-j)})(1-|q|^{2(\ell+i)})}{(1-|q|^{2(2\ell)})(1-|q|^{2(2\ell+1)})}}.$$

Theorem

$$\beta_+^+(\ell,i,j) = -q^{j-i-1}(\overline{q})^{\ell-j+1}\sqrt{\frac{q}{\overline{q}}}\sqrt{\frac{(1-|q|^{2(\ell-j+1)})(1-|q|^{2(\ell+i+1)})}{(1-|q|^{2(2\ell+1)})(1-|q|^{2(2\ell+2)})}}\,.$$

$$\beta_-^+(\ell,i,j) = (\overline{q})^{\ell-j}\sqrt{\frac{(1-|q|^{2(\ell+j)})(1-|q|^{2(\ell-i)})}{(1-|q|^{2(2\ell)})(1-|q|^{2(2\ell+1)})}}\,.$$

$$\alpha_+(\ell,i,j) = \sqrt{\frac{(1-|q|^{2(\ell-j+1)})(1-|q|^{2(\ell-i+1)})}{(1-|q|^{2(2\ell+1)})(1-|q|^{2(2\ell+2)})}}\,.$$

$$\alpha_-(\ell,i,j) = q^{\ell-i}(\overline{q})^{\ell-j+1}\sqrt{\frac{q}{\overline{q}}}\sqrt{\frac{(1-|q|^{2(\ell+j)})(1-|q|^{2(\ell+i)})}{(1-|q|^{2(2\ell)})(1-|q|^{2(2\ell+1)})}}\,.$$

The Dirac operator

Let $\mathscr{H} = L^2(h) \otimes \mathbb{C}^2$, $\pi_{eq}(x) = \begin{bmatrix} \pi(x) & 0 \\ 0 & \pi(x) \end{bmatrix}$ where π is the GNS representation of $\mathscr{A}_q$ on $L^2(h)$. Also, let

$$\mathscr{D} = \begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix} \quad \text{and} \quad \gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where T is the following unbounded operator on $L^2(h)$ with dense domain $\mathscr{A}_q$ defined by

$$T(e^{\ell}_{i,j,k}) = d(\ell, i, k) e^{\ell}_{i,j,k}$$

such that

$$d(\ell, i, k) = \begin{cases} (2\ell + 1) + \sqrt{-1}(k - \ell - i) & \text{if } \; i \neq -\ell, \\ -(2\ell + 1) + \sqrt{-1}k & \text{if } \; i = -\ell. \end{cases}$$

Equivariant spectral triple

Theorem

The tuple $(\mathscr{A}_q, \mathscr{H}, \pi_{eq}, \mathscr{D}, \gamma)$ *is a* 4^+*-summable, non-degenerate, even spectral triple for* $U_q(2)$ *that is equivariant under its own comultiplication action.*

Theorem

The Chern character of the spectral triple $(\mathscr{A}_q, \mathscr{H}, \pi_{eq}, \mathscr{D}, \gamma)$ *is nontrivial.*

Proof: We are interested in

$$\langle [p \otimes p_\theta], (\mathscr{A}_q, \mathscr{H}, \pi_{eq}, \mathscr{D}, \gamma) \rangle = ind((p \otimes p_\theta) T |T|^{-1} (p \otimes p_\theta)) .$$

Idea of the proof

1. Let P be the orthogonal projection onto $\{v \in L^2(h) : bb^*(v) = v\}$ and consider the C^*-algebra $\mathcal{B} = C^*\{Pb, PD\}$. Since θ is irrational, this C^*-algebra becomes isomorphic to the noncommutative torus $\mathbb{A}_\theta$, and this helps us to identify $p \otimes p_\theta$ with a projection P_θ in $\mathcal{B}$. Then, $PP_\theta = P_\theta P = P_\theta$.
2. The Fredholm operator $P_\theta T|T|^{-1}P_\theta$ is a compact perturbation of $P_\theta F P_\theta$ for certain operator $F : PL^2(h) \to PL^2(h)$.
3. We decompose $PL^2(h) = \oplus_{j=0}^{\infty}\mathcal{H}_j$ such that

$$P = \oplus_j P_j \ , \ F = \oplus_j F_j \ , \ P_\theta F P_\theta = \oplus_j P_\theta^j F_j P_\theta^j$$

   where $P_\theta^j = P_\theta|_{\mathcal{H}_j}$.
4. It turns out that for $j_1, j_2 > 0$, $ind(P_\theta^{j_1} F_{j_1} P_\theta^{j_1}) = ind(P_\theta^{j_2} F_{j_2} P_\theta^{j_2})$. Moreover, $ind(P_\theta^0 F_0 P_\theta^0) \neq 0$.
5. Finally, we prove that $P_\theta F P_\theta : P_\theta L^2(h) \to P_\theta L^2(h)$ is Fredholm.

References

Satyajit Guin and Bipul Saurabh.
Representations and classification of the compact quantum groups $U_q(2)$ for complex deformation parameters.
Internat. J. Math. 32 (2021), no. 4, 2150020.

Satyajit Guin and Bipul Saurabh.
K-theory and equivariant spectral triple for the quantum group $U_q(2)$ for complex deformation parameters.
arXiv:2102.11473.

Thank You !